THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Tutorial 10
28th November 2024

- Tutorial exercise would be uploaded to blackboard on Mondays provided that there is a tutorial class on that Thursday. You are not required to hand in the solution, but you are advised to try the problems before tutorial classes. Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

- **Important: The extra content of this tutorial will not appear in the final exam.**

Most rings that we will encounter or use are rings with unity, so unless specified otherwise, "a ring" will mean "a ring with unity".

1. Recall from HW10 the definition of a commutative Noetherian ring.

   (a) Prove that a (nonzero) Noetherian ring has a maximal ideal without assuming the axiom of choice.

   (b) Prove that any PID is Noetherian.

   (c) Find a counterexample where a UFD may not be Noetherian.

   (d) Prove that if $R \to R$ is a surjective homomorphism, then it is an isomorphism.

2. A (commutative) Artinian ring is a ring R which satisfies the descending chain condition (DCC): any descending sequence of ideals eventually stabilizes, i.e. if $I_1 \supset I_2 \supset I_3 \supset ...$ is a descending sequence of ideals, then $I_k = I_{k+1}$ for all large k.

   (a) Show that $\mathbb{Z}$ is Noetherian but not Artinian.

   (b) Show that $k[t]$ is Noetherian but not Artinian, here k is a field.

   (c) Show that $k[t]/(t^n)$ is Artinian, here k is a field.

   (d) Prove that an Artinian integral domain is a field.

3. (a) Let $R = \{\sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x] : a_1 = 0\}$ the subring of polynomial ring over the integers where each polynomial has no linear term. Show that R is not a UFD.

   (b) Find an irreducible element in R that is not prime.

   (c) Recall that in $\mathbb{Z}$, we can always write the gcd as $\gcd(x, y) = ax + by$ for some other integers a, b. Prove that it is not true in R, by finding gcd of x^2 and x^3.

4. (a) Recall the definition of gcd: we say d is a gcd of a, b if d divides a and b, and for every d' dividing both a and b, d' also divides d. Given an example to illustrate why two elements may not have a well-defined gcd in a general integral domain.

   (b) Show that for any two elements in a UFD, they have a gcd that is unique up to unit. (Such ring is called a GCD domain, we are proving that UFD are GCD domains.)

   (c) Let D be a UFD, show that $\gcd(a, b)\text{lcm}(a, b) = ab$.

5. Suppose that D is a discrete valuation ring (DVR), i.e. D is a PID with unique maximal ideal $\mathfrak{m}$, show that for any $x \in D \setminus 0$, there is some largest integer k so that $\langle x \rangle \subset \mathfrak{m}^k$. Hence argue that $\nu(x) = k$ defines an Euclidean norm on D.

6. Factorize 13 into a product in the Gaussian integers $\mathbb{Z}[i]$.

7. In the lectures, we have seen the following inclusions

   $$\text{Commutative rings} \supset \text{Integral domains} \supset \text{UFD} \supset \text{PID} \supset \text{Euclidean domains} \supset \text{Fields}$$

   It is a good exercise/revision to try to name counterexamples to demonstrate that the above inclusions are proper.